arXiv:1010.1695v2 [math.DG] 19 Jul 2012

EXCEPTIONAL HOLONOMY ON VECTOR BUNDLES WITH TWO-DIMENSIONAL FIBERS

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Abstract. An $SU(3)$- or $SU(1,2)$-structure on a 6-dimensional manifold N^6 can be defined as a pair of a 2-form ω and a 3-form ρ. We prove that any analytic $SU(3)$- or $SU(1,2)$-structure on N^6 with $d\omega \wedge \omega = 0$ can be extended to a parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure Φ that is defined on the trivial disc bundle $N^6 \times B_\epsilon(0)$ for a sufficiently small $\epsilon > 0$. Furthermore, we show by an example that Φ is not uniquely determined by (ω, ρ) and discuss if our result can be generalized to non-trivial bundles.

Contents

1. Introduction

In his article on stable forms, Hitchin [9] proposed a new method to construct manifolds with exceptional holonomy. The starting point of his construction is a 7-dimensional manifold M with a G_2-structure ϕ that satisfies $d*\phi = 0$. We can take ϕ as an initial value for a certain flow equation such that the solution of the initial value problem yields a parallel Spin(7)-structure on $M \times (-\epsilon, \epsilon)$ for an $\epsilon > 0$. This idea can be generalized to the semi-Riemannian case where we obtain a parallel $\mathrm{Spin}_0(3,4)$-structure [6].

Many of the known complete metrics with holonomy Spin(7) are not defined on a manifold of type $M \times (-\epsilon, \epsilon)$ but on a disc bundle over a lower-dimensional manifold [1, 2, 5, 7, 10, 11, 14]. The reason behind this is that

those metrics are of cohomogeneity one and that the cohomogeneity-one manifolds of this type are the only ones that admit complete metrics with holonomy Spin(7) [14].

Bielawski [3] proves another result that fits into this context. Let X be a real analytic Kähler manifold. We identify X with the zero section of its canonical bundle. The Kähler metric on X can be uniquely extended to a Ricci-flat Kähler metric on a neighborhood of X such that the $U(1)$-action on the bundle is isometric and Hamiltonian. We thus have extended the $U(n)$-structure on the base to an $SU(n+1)$-structure on the bundle.

Motivated by these facts, we attempt to construct parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structures on $\mathbb{R}^2$-bundles. More precisely, let (ω, ρ) be a pair of a 2-form and a 3-form on a 6-dimensional manifold N^6 that defines an $SU(3)$- or $SU(1,2)$-structure. We search for conditions on (ω, ρ) such that on $N^6 \times B_\epsilon(0)$, where $B_\epsilon(0)$ is a ball of radius $\epsilon > 0$ in $\mathbb{R}^2$, there exists a parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure that extends in a suitable sense the G-structure (ω, ρ).

The article is organized as follows. In Section 2 and 3 we give an introduction to the G-structures that we need and to Hitchin's flow equation. We set up our initial value problem and prove that it has a local solution in the following section. After that we show with help of an example that our solution can be non-unique. In the sixth section, we finally discuss if our result can be generalized to non-trivial bundles over 6-dimensional manifolds.

2. G-structures

2.1. **G is $SU(3)$ or $SU(1,2)$.** In order to prove our theorem we have to introduce several G-structures. We start with G-structures on 6-dimensional manifolds and then proceed to the 7- and 8-dimensional case. A well written introduction to all of these G-structures can be found in Cortés et al. [6]. We use similar conventions as [6] and only recapitulate the facts that we need for our considerations. Although a G-structure is in general defined as a principal bundle, all G-structures in this section can be described with help of certain differential forms. Throughout this article we use the following convention.

Convention 2.1. Let $(v_i)_{i\in I}$ be a basis of a vector space V. We denote its dual basis by $(v^i)_{i\in I}$ and abbreviate $v^{i_1} \wedge \ldots \wedge v^{i_k}$ by $v^{i_1\cdots i_k}$.

Let $(e_i)_{i=1,\ldots,6}$ be the canonical basis of $\mathbb{R}^6$. We define the 2-forms

$$\omega_{SU(3)} := e^{12} + e^{34} + e^{56} \tag{1}$$

and

$$\omega_{SU(1,2)} := -e^{12} - e^{34} + e^{56}\,. \tag{2}$$

Moreover, we introduce the canonical 3-form

$$\rho_{can.} := e^{135} - e^{146} - e^{236} - e^{245}\,. \tag{3}$$

The following lemma is proven in [6].

Lemma 2.2. *Let $G \in \{SU(3), SU(1,2)\}$. The subgroup of all $A \in GL(6,\mathbb{R})$ that stabilize ω_G and $\rho_{can.}$ simultaneously is isomorphic to G.*

This motivates the following definition.

Definition 2.3. Let $G \in \{SU(3), SU(1,2)\}$, V be a 6-dimensional real vector space and (ω, ρ) be a pair of a 2-form and a 3-form on V. If there exists a basis $(v_i)_{i=1,\ldots,6}$ of V such that with respect to this basis ω can be identified with ω_G and ρ with $\rho_{can.}$, (ω,ρ) is called a *G-structure*.

Hitchin [9] has introduced the notion of a stable form.

Definition 2.4. Let V be a real or complex vector space and $\beta \in \bigwedge^k V^*$ with $k \in \{0, \ldots, \dim V\}$ be a k-form. β is called *stable* if the $GL(V)$-orbit of β is an open subset of $\bigwedge^k V^*$.

Lemma 2.5. *Let (ω,ρ) be a G-structure where $G \in \{SU(3), SU(1,2)\}$. In this situation, ω and ρ are both stable forms.*

Remark 2.6. The stable forms are an open dense subset of $\bigwedge^2 \mathbb{R}^{6*}$ and of $\bigwedge^3 \mathbb{R}^{6*}$. There is exactly one open $GL(6,\mathbb{R})$-orbit in $\bigwedge^2 \mathbb{R}^{6*}$ and two open orbits in $\bigwedge^3 \mathbb{R}^{6*}$. One of them is the orbit of $\rho_{can.}$. The other one can be used to define the notion of an $SL(3,\mathbb{R})$-structure, which we will not consider in this article.

Let V be a 6-dimensional real vector space and $\bigwedge_s^k V^*$ be the set of all stable k-forms on V. We can assign to any $\rho \in \bigwedge_s^3 V^*$ a certain endomorphism J_ρ by a map

$$i : \bigwedge\nolimits_s^3 V^* \to V \otimes V^*\,. \tag{4}$$

i is a rational $GL(6,\mathbb{R})$-equivariant map and is described in detail in [6]. $i(\rho_{can.})$ is the canonical complex structure on $\mathbb{R}^6$ which maps e_{2i-1} to $-e_{2i}$ and e_{2i} to e_{2i-1} for all $i \in \{1,2,3\}$. If (ω,ρ) is an $SU(3)$- or an $SU(1,2)$-structure, J_ρ is a complex structure, too. With help of another map

$$j : \bigwedge\nolimits_s^2 V^* \times \bigwedge\nolimits_s^3 V^* \to S^2(V^*) \tag{5}$$

we can assign to (ω,ρ) a symmetric non-degenerate bilinear form. j is also a rational $GL(6,\mathbb{R})$-equivariant map that is described explicitly in [6]. If (ω,ρ) is an

(1) $SU(3)$-structure, $j(\omega,\rho)$ is a metric with signature $(6,0)$. In particular, $j(\omega_{SU(3)},\rho_{can.})$ is the Euclidean metric on $\mathbb{R}^6$.
(2) $SU(1,2)$-structure, $j(\omega,\rho)$ is a metric with signature $(2,4)$. In particular,

$$j(\omega_{SU(1,2)},\rho_{can.}) \;=\; -e^1\otimes e^1 - e^2\otimes e^2 - e^3\otimes e^3 - e^4\otimes e^4 + e^5\otimes e^5 + e^6\otimes e^6\,. \tag{6}$$

Convention 2.7. (1) We call J_ρ the *complex structure that is associated to* ρ or shortly the *associated complex structure.*
(2) We call $j(\omega,\rho)$ the *metric that is associated to* (ω,ρ) or shortly the *associated metric*. We denote it by g_6, since we will also work with metrics on 7- or 8-dimensional spaces.

We remark that the objects that we have defined are related by the formula

$$\omega(v,w) := g_6(v,J_\rho(w))\,. \tag{7}$$

We can decide if a pair (ω,ρ) determines an $SU(3)$- or $SU(1,2)$-structure without referring to a special basis.

Theorem 2.8. *Let V be a 6-dimensional real vector space and let $\omega\in\bigwedge^2 V^*$ and $\rho\in\bigwedge^3 V^*$ be stable. Moreover, let J_ρ and g_6 be defined as above. We assume that ω and ρ satisfy the equations*

(1) $\omega\wedge\rho=0$,
(2) $J_\rho^*\rho\wedge\rho=\frac{2}{3}\omega\wedge\omega\wedge\omega$.

If in this situation

(1) g_6 has signature $(6,0)$ and J_ρ is a complex structure, (ω,ρ) is an $SU(3)$-structure.
(2) g_6 has signature $(2,4)$ and J_ρ is a complex structure, (ω,ρ) is an $SU(1,2)$-structure.

Remark 2.9. (1) Since $J_\rho^*\rho\wedge\rho$ and $\frac{2}{3}\omega\wedge\omega\wedge\omega$ are both 6-forms, the second condition from the theorem is a normalization of the pair (ω,ρ).
(2) If (ω,ρ) is a pair of stable forms satisfying $\omega\wedge\rho=0$ and $J_\rho^*\rho\wedge\rho=\frac{2}{3}\omega\wedge\omega\wedge\omega$ and it is not an $SU(3)$- or $SU(1,2)$-structure, J_ρ is a para-complex structure and (ω,ρ) is an $SL(3,\mathbb{R})$-structure.

The reason for the above considerations is to define G-structures on manifolds.

Definition 2.10. Let M be a 6-dimensional manifold, $\omega \in \bigwedge^2 T^*M$, and $\rho \in \bigwedge^3 T^*M$. Moreover, let $G \in \{SU(3), SU(1,2)\}$. (ω, ρ) is called a *G-structure on M* if for all $p \in M$ (ω_p, ρ_p) is a G-structure on T_pM.

Convention 2.11. Since the endomorphism field J_ρ in general has torsion, we call it the *almost* complex structure on M.

2.2. **G is G_2 or G_2^*.** With help of the concepts from the previous subsection we are able to define G_2- and G_2^*-structures.

Definition and Lemma 2.12. We supplement the basis $(e_i)_{i=1,\ldots,6}$ of $\mathbb{R}^6$ with e_7 to a basis of $\mathbb{R}^7$. The form

(1) $\phi_{G_2} := \omega_{SU(3)} \wedge e^7 + \rho_{can.}$ is stabilized by G_2.
(2) $\phi_{G_2^*} := \omega_{SU(1,2)} \wedge e^7 + \rho_{can.}$ is stabilized by G_2^*.

G_2 denotes the compact real form of the complex Lie group $G_2^{\mathbb{C}}$ and G_2^* denotes the split real form. Let V be a 7-dimensional real vector space and ϕ be a 3-form on V. If there exists a basis $(v_i)_{i=1,\ldots,7}$ of V such that with respect to $(v_i)_{i=1,\ldots,7}$

(1) ϕ can be identified with ϕ_{G_2}, ϕ is called a *G_2-structure*.
(2) ϕ can be identified with $\phi_{G_2^*}$, ϕ is called a *G_2^*-structure*.

Remark 2.13. There are exactly two open orbits of the action of $GL(7,\mathbb{R})$ on $\bigwedge^3 \mathbb{R}^{7*}$ [13, 15]. Their union is a dense subset of $\bigwedge^3 \mathbb{R}^{7*}$. One orbit consists of all 3-forms that are stabilized by G_2 and the other one consists of all 3-forms that are stabilized by G_2^*.

Any G_2- or G_2^*-structure on a vector space V determines a symmetric non-degenerate bilinear form g_7 and a volume form vol_7. As in the previous subsection, there are explicit rational $GL(7,\mathbb{R})$-equivariant maps $\bigwedge^3_s V^* \to S^2(V^*)$ and $\bigwedge^3_s V^* \to \bigwedge^7 V^*$ that assign g_7 and vol_7 to ϕ. The explicit definition of these maps can be found in [6]. The tensors ϕ, g_7, and vol_7 are related by the formula

$$g_7(v,w)\,\mathrm{vol}_7 = \tfrac{1}{6}(v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \phi \quad \forall v, w \in V\,. \tag{8}$$

Analogously to Subsection 2.1, we have

Lemma 2.14. *Let V be a 7-dimensional real vector space and ϕ be a stable 3-form on V.*

(1) *If ϕ is a G_2-structure, g_7 has signature $(7,0)$. In particular, g_7 is the Euclidean metric on $\mathbb{R}^7$ if ϕ coincides with ϕ_{G_2}.*
(2) *If ϕ is a G_2^*-structure, g_7 has signature $(3,4)$. In particular, $g_7 = g_6 + e^7 \otimes e^7$ if ϕ coincides with $\phi_{G_2^*}$.*

We can relate vol_7 to the 3-forms on the 6-dimensional subspace $\mathrm{span}(v_i)_{i=1,\ldots,6}$.

Lemma 2.15. *Let ϕ be a G_2- or G_2^*-structure on a vector space V and $(v_i)_{i=1,\ldots,7}$ be a basis of V with the properties from Definition and Lemma 2.12. On $span(v_i)_{i=1,\ldots,6}$ there exists a canonical $SU(3)$- or $SU(1,2)$-structure (ω,ρ) and we have*

$$vol_7 = \tfrac{1}{4} J_\rho^* \rho \wedge \rho \wedge v^7 \ . \tag{9}$$

In particular, vol_7 is $e^{1234567}$ if ϕ is ϕ_{G_2} or $\phi_{G_2^*}$.

g_7 and vol_7 determine a Hodge-star operator $*$ on $\bigwedge^* V^*$.

Lemma 2.16. *Let ϕ be a G_2- or G_2^*-structure. The 4-form $*\phi$ is stable and can be described as*

$$v^7 \wedge J_\rho^* \rho + \tfrac{1}{2}\omega \wedge \omega \ . \tag{10}$$

Convention 2.17. We call g_7 (vol_7, $*\phi$) the *metric (volume form, 4-form) that is associated to ϕ.*

We define G_2- and G_2^*-structures on manifolds as in the previous subsection.

Definition 2.18. Let M be a 7-dimensional manifold and $\phi \in \bigwedge^3 T^*M$. Moreover, let $G \in \{G_2, G_2^*\}$. ϕ is called a *G-structure on M* if for all $p \in M$ ϕ_p is a G-structure on T_pM.

2.3. G is $\mathbf{Spin(7)}$ or $\mathbf{Spin_0(3,4)}$.

In this final subsection, we introduce $\mathrm{Spin}(7)$- and $\mathrm{Spin}_0(3,4)$-structures.

Definition and Lemma 2.19. We supplement the basis $(e_i)_{i=1,\ldots,7}$ of $\mathbb{R}^7$ with e_8 to a basis of $\mathbb{R}^8$. The form

1. $\Phi_{\mathrm{Spin}(7)} := e^8 \wedge \phi_{G_2} + *\phi_{G_2}$ is stabilized by $\mathrm{Spin}(7)$.
2. $\Phi_{\mathrm{Spin}_0(3,4)} := e^8 \wedge \phi_{G_2^*} + *\phi_{G_2^*}$ is stabilized by the identity component $\mathrm{Spin}_0(3,4)$ of $\mathrm{Spin}(3,4)$.

Let V be an 8-dimensional real vector space and Φ be a 4-form on V. If there exists a basis $(v_i)_{i=1,\ldots,8}$ of V such that with respect to $(v_i)_{i=1,\ldots,8}$

1. Φ can be identified with $\Phi_{\mathrm{Spin}(7)}$, Φ is called a *Spin(7)-structure.*
2. Φ can be identified with $\Phi_{\mathrm{Spin}_0(3,4)}$, Φ is called a *$Spin_0(3,4)$-structure.*

Analogously to Subsection 2.1 and 2.2, any $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure determines a symmetric non-degenerate bilinear form g_8 and a volume form vol_8. vol_8 is given by $\frac{1}{14}\Phi \wedge \Phi$ and g_8 satisfies a slightly more complicated relation as (8), which can be found in Karigiannis [12].

Unlike ω, ρ, and ϕ, Φ is not a stable form. Nevertheless, we have similar results as in the previous two subsections.

Lemma 2.20. *Let Φ be a Spin(7)- or $Spin_0(3,4)$-structure. In the first case g_8 has signature $(8,0)$ and in the second case it has signature $(4,4)$. In particular, g_8 is the Euclidean metric on $\mathbb{R}^8$ if Φ coincides with $\Phi_{Spin(7)}$ and $g_8 = g_7 + e^8 \otimes e^8$ if Φ coincides with $\Phi_{Spin_0(3,4)}$. In both cases, we have*

$$vol_8 = vol_7 \wedge v^8 \ . \tag{11}$$

Convention 2.21. As in the previous subsections, we call g_8 the *associated metric* and vol_8 the *associated volume form*.

Remark 2.22.
(1) Φ is self-dual with respect to g_8 and vol_8.
(2) Any 4-form on an 8-dimensional real vector space that is stabilized by Spin(7) or $\mathrm{Spin}_0(3,4)$ is a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure. However, there is no simple criterion like Theorem 2.8 that decides if a given 4-form is a Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure.

The notion of a Spin(7)- or a $\mathrm{Spin}_0(3,4)$-structure on an 8-dimensional manifold can be defined completely analogously to Definition 2.10 and 2.18.

3. Hitchin's flow equation

One motivation to study G-structures is their relation to metrics with special holonomy.

Definition 3.1. Let $G \in \{\mathrm{Spin}(7), \mathrm{Spin}_0(3,4)\}$ and let Φ be a G-structure on an 8-dimensional manifold. Φ is called *torsion-free* if $d\Phi = 0$.

Lemma 3.2. *Let G be as above. The holonomy group of the metric that is associated to a torsion-free G-structure is a subgroup of G. Conversely, let (M,g) be a semi-Riemannian manifold whose holonomy is contained in G. Then there exists a torsion-free G-structure on M whose associated metric is g.*

Proof. See [8] for $G = \mathrm{Spin}(7)$ and [4] for $G = \mathrm{Spin}_0(3,4)$. □

Remark 3.3. There are analogous results for $G \in \{SU(3), SU(1,2), G_2, G_2^*\}$.

We also need the following G-structures with torsion.

Definition 3.4. Let ϕ be a G_2- or G_2^*-structure on a 7-dimensional manifold. ϕ is called *cocalibrated* if $d * \phi = 0$.

Compact Riemannian manifolds with holonomy Spin(7) are hard to construct. However, many non-compact examples with cohomogeneity one are known [1, 2, 5, 7, 10, 11, 14]. All of the these metrics can be obtained by a method that was developed by Hitchin [9]. As in the previous section, our presentation of the issue is similar as in [6].

Theorem 3.5. *(See [6, 9]) Let N^7 be a 7-dimensional manifold and $U \subset N^7 \times \mathbb{R}$ be an open neighborhood of $N^7 \times \{0\}$. Furthermore, let $G \in \{G_2, G_2^*\}$ and ϕ be a cocalibrated G-structure on N^7. Finally, let ϕ_t be a one-parameter family of 3-forms such that ϕ_t is defined on $U \cap (N^7 \times \{t\})$. We assume that ϕ_t is a solution of the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} *_7 \phi_t &= d_7 \phi_t \\ \phi_0 &= \phi \end{aligned} \tag{12}$$

The index "7" emphasizes that we consider $*$ and d as operators on $U \cap (N^7 \times \{t\})$ instead of U. If U is sufficiently small, ϕ_t is a G-structure for all t with $U \cap (N^7 \times \{t\}) \neq \emptyset$. Moreover, it is cocalibrated for all t. The 4-form

$$\Phi := dt \wedge \phi_t + *_7 \phi_t \tag{13}$$

is a torsion-free Spin(7)-structure if $G = G_2$ and a torsion-free $Spin_0(3,4)$-structure if $G = G_2^*$. Let g_8 be the metric that is associated to Φ and g_t be the metric on $N^7 \times \{t\}$ that is associated to ϕ_t. With this notation we have

$$g_8 = g_t + dt^2 \, . \tag{14}$$

Remark 3.6.

1. The equation $\frac{\partial}{\partial t} *_7 \phi_t = d_7 \phi_t$ is called *Hitchin's flow equation*. Since $*_7$ depends non-linearly on ϕ_t, it is a non-linear partial differential equation.
2. If N^7 and ϕ_0 are real analytic, the system (12) has a unique maximal solution that is defined on an open neighborhood of $N^7 \times \{0\}$ [6]. This is a consequence of the Cauchy-Kovalevskaya Theorem. We assume from now that all initial data are analytic.
3. If N^7 is in addition compact, there exists a unique maximal open interval I with $0 \in I$ such that the solution is defined on $N^7 \times I$.
4. Let $f : N^7 \to N^7$ be a diffeomorphism, I an interval with $0 \in I$, $U = N^7 \times I$, and ϕ_t be a solution of Hitchin's flow equation on U. In this situation, the pull-back $f^* \phi_t$ is also a solution with the initial value $f^* \phi_0$.

4. Proof of the main theorem

In this section, we consider a 6-dimensional manifold N^6 that carries an $SU(3)$- or $SU(1,2)$-structure (ω_0, ρ_0). Our aim is to construct a parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure Φ on a tubular neighborhood of the zero section of the trivial bundle $N^6 \times \mathbb{R}^2$ such that the restriction of Φ to N^6 is (ω_0, ρ_0) in a suitable sense. More precisely, let $\epsilon > 0$ be sufficiently small and

$$B_\epsilon(0) := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < \epsilon\}\,. \tag{15}$$

We denote $N^6 \times \{0\} \subset N^6 \times B_\epsilon(0)$ shortly by N^6. On that submanifold we want to have

$$\Phi = \tfrac{1}{2}\omega_0 \wedge \omega_0 + dx \wedge \rho_0 + dy \wedge J^*_{\rho_0}\rho_0 + dx \wedge dy \wedge \omega_0 \tag{16}$$

or equivalently

$$\begin{array}{rcl} \frac{\partial}{\partial y} \lrcorner \left(\frac{\partial}{\partial x} \lrcorner \Phi\right) & = & \omega_0 \\ \frac{\partial}{\partial x} \lrcorner \Phi - dy \wedge \omega_0 & = & \rho_0 \end{array} \tag{17}$$

Our first step is to construct a G_2- or G_2^*-structure ϕ on

$$V_\epsilon := N^6 \times \{(0,y) \in \mathbb{R}^2 | y^2 < \epsilon\}$$

that satisfies

$$\phi = \rho + dy \wedge \omega \quad \text{and} \quad d * \phi = 0 \tag{18}$$

for a y-dependent $SU(3)$- or $SU(1,2)$-structure (ω, ρ) on N^6. Next, we insert ϕ as initial condition into Hitchin's flow equation, where x plays the role of the coordinate t in Theorem 3.5. After that, we have finally found our Φ. We describe how to construct the 3-form on V_ϵ. The Hodge dual of ϕ is

$$* \phi = \frac{1}{2}\omega \wedge \omega + dy \wedge J^*_\rho \rho\,. \tag{19}$$

ϕ is thus cocalibrated if and only if

$$\begin{array}{rcl} \left(\frac{\partial}{\partial y}\omega\right) \wedge \omega & = & dJ^*_\rho \rho \\ d\omega \wedge \omega & = & 0 \end{array} \tag{20}$$

for all y. In the above equation, d denotes the exterior derivative on the 6-dimensional manifold $N^6 \times \{(0,y)\}$. We see that any choice of ρ satisfies the system (20). Since

$$(\omega \wedge \omega)_y = \omega_0 \wedge \omega_0 + 2\int_0^y dJ^*_\rho \rho \, d\widetilde{y} \tag{21}$$

and $d^2 = 0$, $d\omega \wedge \omega = 0$ is satisfied for all y if it is satisfied for $y = 0$. Of course, (ω, ρ) shall be an $SU(3)$- or $SU(1,2)$-structure for all $y \in (-\epsilon, \epsilon)$. Therefore, the system that (ω, ρ) has to satisfy is in fact

$$\begin{aligned}\left(\frac{\partial}{\partial y}\omega\right) \wedge \omega &= dJ_\rho^* \rho \\ \omega \wedge \rho &= 0 \\ 2\,\omega^3 &= 3\,\rho \wedge J_\rho^* \rho\end{aligned} \tag{22}$$

If we take the derivative of the last two equations with respect to y, we obtain the following system of first order differential equations

$$\begin{aligned}\left(\frac{\partial}{\partial y}\omega\right) \wedge \omega &= dJ_\rho^* \rho \\ \left(\frac{\partial}{\partial y}\rho\right) \wedge \omega + \rho \wedge \left(\frac{\partial}{\partial y}\omega\right) &= 0 \\ 3\left(\frac{\partial}{\partial y}\rho\right) \wedge J_\rho^*\rho + 3\rho \wedge \left(\frac{\partial}{\partial y}J_\rho^*\rho\right) - 6\left(\frac{\partial}{\partial y}\omega\right) \wedge \omega^2 &= 0\end{aligned} \tag{23}$$

with the initial conditions

$$\begin{aligned}d\omega_0 \wedge \omega_0 &= 0 \\ \omega_0 \wedge \rho_0 &= 0 \\ 2\omega_0^3 &= 3\rho_0 \wedge J_{\rho_0}^* \rho_0\end{aligned} \tag{24}$$

Since all forms in a neighborhood of ω_0 or ρ_0 are stable, any solution of (23) and (24) describes a G_2- or G_2^*-structure if ϵ is sufficiently small. Let $z^1, \ldots, z^6$ be coordinates on an open subset $U \subset N^6$. The system (23) consists of 22 equations for the 35 coefficient functions of ω and ρ. It can be written as

$$F\left(\omega, \rho, \frac{\partial\omega}{\partial z^1}, \ldots, \frac{\partial\omega}{\partial z^6}, \frac{\partial\rho}{\partial z^1}, \ldots, \frac{\partial\rho}{\partial z^6}, \frac{\partial\omega}{\partial y}, \frac{\partial\rho}{\partial y}\right) = 0\,. \tag{25}$$

ω is up to the sign uniquely determined by ω^2 [6, 9]. The first equation of (23) thus fixes the value of $\frac{\partial\omega}{\partial y}$. The second and third equation restrict ρ at each $p \in U$ to a submanifold of $\bigwedge^3 T_p^* U$ of codimension 7. $(dF)_{\left(\frac{\partial\omega}{\partial y}, \frac{\partial\rho}{\partial y}\right)}$ therefore has maximal rank. The metric that is associated to (ω, ρ) induces a metric on $\bigwedge^3 T_p^* U$. We denote the orthogonal projection of a 3-form to the tangent space of the set of all ρ that satisfy $\omega \wedge \rho = 0$ and $2\omega^3 = 3\rho \wedge J_\rho^* \rho$ by π_ω. We add the equation

$$\pi_\omega\left(\frac{\partial\rho}{\partial y}\right) = 0 \tag{26}$$

to (23) and obtain a system of type (25), where F is replaced by a an $\widetilde{F}$ that satisfies

$$\operatorname{rk}(d\widetilde{F})_{(\frac{\partial\omega}{\partial t},\frac{\partial\rho}{\partial t})} = 35\,. \tag{27}$$

With help of the implicit function theorem, the extended system can be rewritten to

$$\begin{aligned} \frac{\partial\omega}{\partial y} &= F_1\left(\omega,\rho,\frac{\partial\omega}{\partial x^1},\ldots,\frac{\partial\omega}{\partial x^6},\frac{\partial\rho}{\partial x^1},\ldots,\frac{\partial\rho}{\partial x^6}\right)\\ \frac{\partial\rho}{\partial y} &= F_2\left(\omega,\rho,\frac{\partial\omega}{\partial x^1},\ldots,\frac{\partial\omega}{\partial x^6},\frac{\partial\rho}{\partial x^1},\ldots,\frac{\partial\rho}{\partial x^6}\right)\end{aligned} \tag{28}$$

Since N^6 is a real analytic manifold, F_1 and F_2 are analytic, too. As in [6], the Cauchy-Kovalevskaya theorem guarantees that the extended system has a unique solution on an open neighbourhood of $N^6 \subset N^6 \times \mathbb{R}$. Thus, (23) has at least one solution on the same open set. If N^6 is compact, the solution exists on V_ϵ for a sufficiently small $\epsilon > 0$. With help of Theorem 3.5, we are finally able to prove our main theorem.

Theorem 4.1. *Let N^6 be an analytic compact 6-manifold and let (ω_0,ρ_0) be an analytic $SU(3)$- or $SU(1,2)$-structure with $d\omega_0\wedge\omega_0 = 0$ on N^6. Then, there exists an $\epsilon > 0$ and a parallel Spin(7)- or Spin$_0(3,4)$-structure Φ on $N^6 \times B_\epsilon(0)$ such that on $N^6\times\{0\}$ we have*

$$\begin{aligned} \frac{\partial}{\partial y}\lrcorner\frac{\partial}{\partial x}\lrcorner\Phi &= \omega_0\\ \frac{\partial}{\partial x}\lrcorner\Phi - dy\wedge\omega_0 &= \rho_0\end{aligned} \tag{29}$$

where x and y are the standard coordinates on $B_\epsilon(0)$.

5. An example

In this section, we show that the 4-form Φ from Theorem 4.1 is not uniquely determined by the initial value (ω_0,ρ_0). Before we start, we define what we mean by uniqueness in this situation.

Definition 5.1. Let Φ_1 and Φ_2 be two Spin(7)- or Spin$_0(3,4)$-structures on $N^6 \times B_\epsilon(0)$ such that on $N^6\times\{0\}$ we have

$$\begin{aligned} \frac{\partial}{\partial y}\lrcorner\frac{\partial}{\partial x}\lrcorner\Phi_1 &= \frac{\partial}{\partial y}\lrcorner\frac{\partial}{\partial x}\lrcorner\Phi_2 &&=: \omega_0\\ \frac{\partial}{\partial x}\lrcorner\Phi_1 - dy\wedge\omega_0 &= \frac{\partial}{\partial x}\lrcorner\Phi_2 - dy\wedge\omega_0 \end{aligned} \tag{30}$$

We call Φ_1 and Φ_2 *equivalent* if there exists a diffeomorphism f of $N^6\times B_\epsilon(0)$ that is the identity on $N^6\times\{0\}$ and satisfies $f^*\Phi_1 = \Phi_2$. Analogously, let

ϕ_1 and ϕ_2 be G_2- or G_2^*-structures on $N^6 \times (-\epsilon, \epsilon)$ such that on $N^6 \times \{0\}$ we have

$$
\begin{aligned}
\tfrac{\partial}{\partial y} \lrcorner \phi_1 &= \tfrac{\partial}{\partial y} \lrcorner \phi_2 &&=: \omega_0 \\
\phi_1 - dy \wedge \omega_0 &= \phi_2 - dy \wedge \omega_0 &&
\end{aligned}
\tag{31}
$$

ϕ_1 and ϕ_2 are called *equivalent* if there exists a diffeomorphism of $N^6 \times (-\epsilon, \epsilon)$ with the same properties as above.

We restrict ourselves to the Riemannian case. For our example, (ω_0, ρ_0) shall be torsion-free. In other words, N^6 together with the initial $SU(3)$-structure is a Calabi-Yau manifold. Our strategy is to construct a one-parameter family of G_2-structures ϕ_δ on $N^6 \times S^1$ such that the standard coordinate $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of S^1 plays the role of y. After that, we consider Hitchin's flow equation with initial value ϕ_δ in order to obtain 4-forms Φ_δ. Let α be a closed 3-form on N^6. We define a G_2-structure ϕ_δ on $N^6 \times S^1$ by

$$
\phi_\delta = \omega_0 \wedge d\theta - J_{\rho_0}^* \rho_0 + \delta \cdot \sin\theta \cdot *_6 \alpha \,, \tag{32}
$$

where $*_6$ is the Hodge-star on N^6. We have

$$
* \phi_\delta = d\theta \wedge (\rho_0 + \delta \cdot \sin\theta \cdot \alpha) + \tfrac{1}{2} \omega_0 \wedge \omega_0 \,. \tag{33}
$$

Since ϕ_0 is a G_2-structure, ϕ_δ is also a G_2-structure if δ is sufficiently small. Moreover, ϕ_δ is cocalibrated and at $\theta = 0$ each term of (31) is independent of δ. Let g_6 be the metric on N^6 that is associated to (ω_0, ρ_0) and $g_{8,\delta}$ be the metric on $N^6 \times S^1 \times (-\epsilon, \epsilon)$ that is associated to Φ_δ. Since ϕ_0 and Φ_0 are both torsion-free, we have $g_{8,0} = g_6 + d\theta^2 + dx^2$ and the second fundamental form II of $N^6 \times \{(0,0)\}$ vanishes. If we find an α such that $II \neq 0$, Φ_0 and Φ_δ are non-equivalent.

Let X be a unit vector field on N^6. X can be lifted to a vector field on the product $N^6 \times S^1 \times (-\epsilon, \epsilon)$. Outside of $N^6 \times \{(0,0)\}$, X is in general not a unit vector field anymore. For all α, $\frac{\partial}{\partial\theta}$ is a unit normal field of $N^6 \times \{(0,0)\}$. Since $[X, \frac{\partial}{\partial\theta}] = 0$, we have on $N^6 \times \{(0,0)\}$

$$
\begin{aligned}
g\left(II(X,X), \tfrac{\partial}{\partial\theta}\right) &= g\left(\nabla_X X, \tfrac{\partial}{\partial\theta}\right) \\
&= \tfrac{1}{2}\left(Xg(X, \tfrac{\partial}{\partial\theta}) + Xg(\tfrac{\partial}{\partial\theta}, X) - \tfrac{\partial}{\partial\theta} g(X,X)\right) \\
&= -\tfrac{1}{2} \tfrac{\partial}{\partial\theta} g(X,X) \,.
\end{aligned}
\tag{34}
$$

Since we can prescribe the value of a closed 3-form at a fixed point arbitrarily, there exists an α such that the last term of the above equation does not

vanish globally if $\delta > 0$. We thus have proven that Φ_0 and Φ_δ are non-equivalent, although they share the same initial values.

6. Outlook

Let N^6 be a 6-dimensional manifold and M^8 be an arbitrary $\mathbb{R}^2$-bundle over N^6. For reasons of brevity, we denote the zero section of M^8 also by N^6. We check under which conditions M^8 admits a not necessarily parallel Spin(7)- or $\mathrm{Spin}_0(3,4)$-structure Φ.

First, we assume that a Spin(7)-structure Φ exists on M^8. Let $\pi : M^8 \to N^6$ be the projection map and $\pi^{-1}(U)$ with $U \subset N^6$ be the image of a local trivialization. Moreover, let e_x and e_y be orthonormal vertical vector fields on $\pi^{-1}(U)$ and (e^x, e^y) be the duals of (e_x, e_y) with respect to the metric. If we replace in equation (17) $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ by (e_x, e_y) and dy by e^y, we obtain an $SU(3)$-structure (ω, ρ) on U. However, the $SU(3)$-structure can in general not be extended to all of N^6, since the bundle may not admit two global linearly independent sections.

Spin(7) acts transitively on the set of all oriented 6-dimensional subspaces of $\mathbb{R}^8$. The subgroup that fixes a subspace is isomorphic to $U(3)$. Therefore, any 6-dimensional oriented submanifold of a Spin(7)-manifold carries a canonical $U(3)$-structure and this is the most natural kind of geometry to suppose on N^6. In terms of tensor fields, a $U(3)$-structure is defined by a non-degenerate 2-form ω, a Riemannian metric g and an almost complex structure J such that $\omega(X,Y) = g(X, J(Y))$ for all vector fields X and Y. In our situation, the $U(3)$-structure is determined by $\omega := e_y \lrcorner e_x \lrcorner \Phi$ and the restriction of the associated metric to the tangent space of N^6. Our definition of ω is independent of the choice of (e_x, e_y) and ω is thus globally defined. The $\mathrm{Spin}_0(3,4)$-case is completely analogous, since $\mathrm{Spin}_0(3,4)/U(1,2)$ is the Grassmannian of all positive oriented planes in $\mathbb{R}^{4,4}$.

We return to the local situation. The restriction of the 4-form to the subset U of the zero section can be written as

$$\Phi = \tfrac{1}{2}\omega \wedge \omega + e^x \wedge \rho + e^y \wedge J_\rho^* \rho + e^x \wedge e^y \wedge \omega\,. \tag{35}$$

We choose another $\pi^{-1}(\widetilde{U})$ and vertical vector fields $\widetilde{e}_x$ and $\widetilde{e}_y$ on $\widetilde{U}$ with the same properties as above. Moreover, we assume that $U \cap \widetilde{U} \neq \emptyset$. On $\widetilde{U}$ we have

$$\Phi = \tfrac{1}{2}\widetilde{\omega} \wedge \widetilde{\omega} + \widetilde{e}^x \wedge \widetilde{\rho} + \widetilde{e}^y \wedge J_{\widetilde{\rho}}^* \widetilde{\rho} + \widetilde{e}^x \wedge \widetilde{e}^y \wedge \widetilde{\omega} \tag{36}$$

for another $SU(3)$- or $SU(1,2)$-structure $(\widetilde{\omega}, \widetilde{\rho})$. On the intersection $\pi^{-1}(U \cap \widetilde{U})$ we have

$$
\begin{aligned}
\widetilde{e}_x &= \cos\theta\, e_x + \sin\theta\, e_y \\
\widetilde{e}_y &= -\sin\theta\, e_x + \cos\theta\, e_y
\end{aligned} \tag{37}
$$

for a function $\theta : U \cap \widetilde{U} \to \mathbb{R}$. Both terms for Φ coincide only if

$$
\begin{aligned}
\widetilde{\rho} &= \cos\theta\, \rho + \sin\theta\, J^*_\rho \rho \\
J^*_{\widetilde{\rho}} \widetilde{\rho} &= -\sin\theta\, \rho + \cos\theta\, J^*_\rho \rho
\end{aligned} \tag{38}
$$

The transition functions for the bundle M^8 thus have to be transition functions for the bundle $\bigwedge^{3,0} T^* N^6$, too. In other words, M^8 has to be isomorphic to the canonical bundle of N^6 with respect to the almost complex structure J.

Conversely, we assume that M^8 is isomorphic to $\bigwedge^{3,0} T^* N^6$ and that N^6 carries a $U(3)$- or $U(1,2)$-structure (ω, g, J). We choose local trivializations $\varphi_\alpha : U_\alpha \times \mathbb{R}^2 \to \pi^{-1}(U_\alpha)$ such that the transition functions have values in $SO(2)$. Let x and y be the standard coordinates of $\mathbb{R}^2$. There exist unique one-forms e^1 and e^2 such that $\varphi^*_\alpha(e^1) = dx$ and $\varphi^*_\alpha(e^2) = dy$. If the U_α are sufficiently small, there exists a $(3,0)$-form ρ on U_α such that (ω, ρ) is an $SU(3)$- or $SU(1,2)$-structure whose associated metric and almost complex structure coincide with g and J. Any other $(3,0)$-form with the same properties as ρ can be written as

$$
\cos\theta_\alpha\, \rho + \sin\theta_\alpha\, J^* \rho \tag{39}
$$

for a function $\theta_\alpha : U_\alpha \to \mathbb{R}$. We define a 4-form

$$
\Phi = \tfrac{1}{2} \pi^*\omega \wedge \pi^*\omega + e^1 \wedge \pi^*\rho + e^2 \wedge \pi^* J^* \rho + e^1 \wedge e^2 \wedge \pi^*\omega \tag{40}
$$

on $\pi^{-1}(U_\alpha)$. Φ is a $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure. By a similar argument as above, we can prove that Φ is globally defined. The above observations yield the following lemma.

Lemma 6.1. *Let M^8 be an $\mathbb{R}^2$-bundle over a manifold N^6 that admits a $U(3)$- or $U(1,2)$-structure (ω, g, J). M^8 admits a Spin(7)- or Spin$_0(3,4)$-structure if and only if M^8 is isomorphic to the canonical bundle of N^6.*

We therefore propose the following conjecture.

Conjecture 6.2. *Let N^6 be a 6-dimensional manifold with a $U(3)$- or $U(1,2)$-structure (ω, g, J) that satisfies $d\omega \wedge \omega = 0$. Then there exists a parallel Spin(7)- or Spin$_0(3,4)$-structure Φ on a tubular neighboorhood of the zero section of the canonical bundle of N^6 such that*

(1) *the restriction of the associated metric to N^6 coincides with g and*

(2) *$e_y \lrcorner (e_x \lrcorner \Phi) = \omega$ for any two orthonormal vertical vector fields e_x and e_y along N^6.*

We finally remark that unlike in [3] we cannot make Φ unique by assuming that the standard $U(1)$-action on the canonical bundle leaves Φ invariant. Any $U(1)$-action that acts as the identity on the base has a differential of type $A_\theta := \mathrm{diag}(e^{i\theta}, 1, 1, 1)$. The fact that this matrix commutes with $SU(4)$ allows the existence of a $U(1)$-invariant $SU(4)$-structure on the bundle. It is essential for the construction of Bielwaski [3] that this works in any complex dimension. Unfortunately, A_θ does not commute with $\mathrm{Spin}(7)$ or $\mathrm{Spin}_0(3,4)$ if we interpret it as a real 8×8-matrix. Therefore, the $U(3)$- or $U(1,2)$-structure can in general not be extended to a $U(1)$-invariant parallel $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure.

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